THE UNIVERSE IN THE CLASSROOM

No. 91 • Spring 2016 www.astrosociety.org/uitc

390 Ashton Avenue, San Francisco, CA 94112

Cosmic Geometry and the Curvature of Space

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Introduction

The notion that space is curved is a challengingly abstract idea. While students have a strong intuition for the concept of curvature, their intuition is based on the perspective of standing apart from curved objects and viewing them from above, which we cannot do when studying our own space. Instead, we must rely on mathematical descriptions of curvature.

Yet these mathematical descriptions are powerful, and the curvature of space plays a central role in contemporary astronomy and physics. Astronomers regularly measure the curvature of space as both a goal in itself and as a means to learn about the contents of the universe. Much of our present understanding, as well as some of our most mysterious unanswered questions, have their source in these measurements.

To build students' comfort with abstract descriptions of curvature, concrete examples like the surface of the Earth or everyday curved objects are effective stepping stones. Students can explore such objects to connect their intuition with the mathematical description, and even employ the same methods on everyday objects that astronomers use to measure the curvature of space. This provides an opportunity for students to make a hands-on connection with both contemporary science and its underlying mathematics.

Curvature of Space vs Curvature of the Earth

The word *space* has several distinct meanings, so let us first make clear what is meant in the phrase *the curvature of space*. In this context, scientists do not use the word *space* in the sense of *outer space*, but rather *space* here refers to the vast collection of all possible locations that objects can occupy in the universe. *Space* is the name for everywhere that we or any object can possibly go. It is thus impossible to leave space or to even comprehend what it would mean to do so. Any property of space that we want to study, including curvature, must be studied entirely from the inside.

This contrasts with the most familiar curved surface in our lives: the Earth. Today, the most direct way to verify that the Earth is indeed curved is to view it from afar, as in Figure 1. Yet long before it was feasible to do this, ancient scientists were aware of the curvature of the Earth. Our situation when studying space from within is analogous to their challenge, which was to study the curvature of the Earth without being able to leave its surface. They remind us of the power of

Figure 1: This image, known as the Blue Marble, was taken by the astronauts of Apollo 17 while traveling to the Moon. This photograph itself has a fascinating history. It is the first full image of the Earth taken from space, and the only one taken by a human being holding a camera. To learn more, visit http://www.theatlantic.com/technology/archive/2011/04/the-blue-marble-shot-our-first-complete-photograph-of-earth/237167/. *[NASA]*

measurement to reveal what cannot be literally seen: 2,200 years before astronauts snapped the photograph in Figure 1, Eratosthenes in Egypt measured the radius of the Earth to within 15% of the modern value, as depicted in Figure 2.

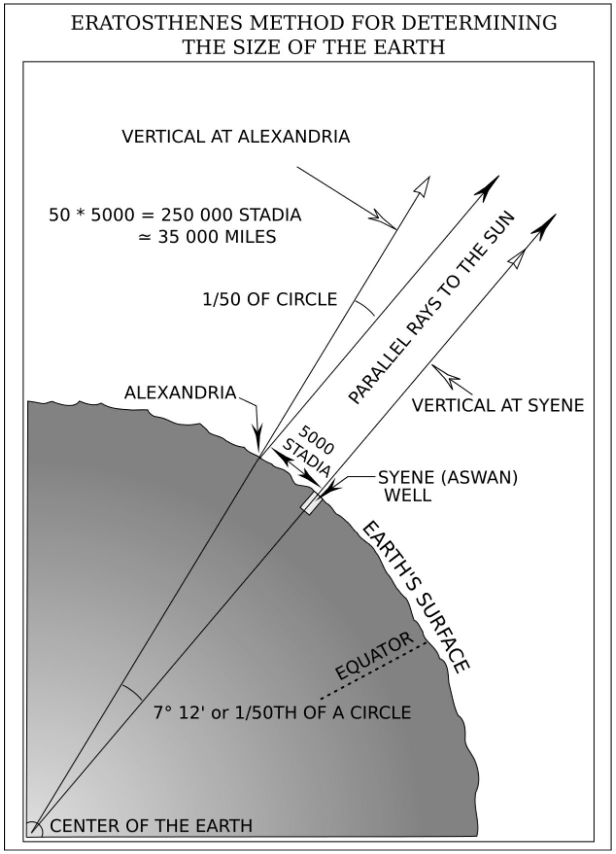

Figure 2: This illustration depicts Eratosthenes's measurement of the curvature of the Earth. Shown is a portion of the globe under the African continent, with sunbeams shown as two rays hitting the ground at Syene and Alexandria. When the sun is directly overhead in Syene, it is not directly overhead in Alexandria due to the curvature of the Earth. By measuring the angle between sunbeams and a vertical stick in Alexandria while the sun was directly overhead in Syene, Eratosthenes was able to estimate the radius of the Earth. This measurement can be replicated in the classroom, as discussed in the activities following the article. To learn more about Eratosthenes, his measurement of the Earth's curvature, and his other achievements, see https://www.khanacademy.org/partner-content/big-history-project/solar-system-and-earth/knowing-solar-system-earth/a/eratosthenes-of-cyrene. *[Wikipedia, Public Domain]*

Curvature from Within

Compared to the long history of terrestrial curvature, the mathematical tools that today allow us to understand the curvature of space are quite young. In 1818, the German mathematician Carl Gauss was spending much of his time focused on the Earth's surface, while also pondering geometry. Gauss was performing a geological survey of the newly created Kingdom of Hanover in present-day Germany, and was busy measuring the distances and angles between important landmarks. At some point, Gauss made an important realization: the curvature of the Earth would cause discrepancies in his measurements, which were based on plane geometry. He recognized that the conventional theorems of geometry are not absolutely true, but are only valid on a flat surface. On a curved surface, the laws of geometry are different.

A parable will help make these ideas intuitive. Suppose there is a construction site, and on the site lives a snake. This is a curious snake who loves geometry, and naturally goes by the name Euclid. Euclid's eyesight is poor and always has been — he cannot see anything but the ground directly in front of him and has absolutely no concept of *up* or *down*. All Euclid knows is *left*, *right*, *forward*, and *back*. Nevertheless, he has a great sense of direction and has learned to measure distances by counting the time it takes to slither between two locations. Before construction begins, Euclid's lot is perfectly flat. For entertainment, Euclid likes to slither back and forth across his lot. If he wants to take the shortest route from one side to the other, what will he do? He will simply travel across the lot in a straight line.

One day workers arrive and dig a deep, smooth hole in the center of the lot. If on the next day Euclid again slithers across, what will he find? It will now take longer to travel directly across, as he must climb into and then out of the hole. But remember, with no understanding of elevation and poor eyesight, Euclid will not recognize his up-and-down motion. Instead, he simply concludes that the distance across his lot is now larger. If he experiments with other routes, he will find the shortest path to the other side of the lot is now a path which curves around the hole. But again, as he traverses this shortest path Euclid will have no sense of avoiding a hole. He simply notices the shortest route is no longer a direct line.

We usually assume that the shortest path between points is a direct line, but Euclid no longer lives on a flat surface and his geometry reflects that. The shortest path between two points is called a *geodesic*, and it is only on a flat surface that geodesics are direct lines. This is one important manifestation of the effect of curvature on geometry, and students can explore it directly by constructing geodesics on curved objects, as discussed in the activity at the end of the article.

Measuring Curvature

Euclid the snake was stuck on a two-dimensional surface and was only aware of four independent ways to move: forward, back, right, and left. We have a very similar situation. We are stuck in a three-dimensional space and know of exactly six independent ways to move: up, down, forward, back, left, and right. Euclid could conclude that something odd had happened to his surface when he found that geodesics were no longer direct lines. We can do the same: simply choose two spots in the universe and measure the length of various paths between them. It may be that a direct line through space is not the shortest route, in which case we draw inspiration from the story of Euclid and say that our space is *curved*. And further, we can characterize the curvature by determining precisely how the geodesics differ from direct lines.

You may have noticed a difficulty: it is not quite feasible to launch a crew of astronauts with tape measures to find geodesics through space whenever we are curious about its curvature. Luckily, we instead have an automatic tracer of geodesics: rays of light. Light always follows the shortest path through empty space, and so if we see light traveling in a direct line, then we know it is crossing a region of flat space. On the other hand, when light bends we know it must be crossing a region of curved space. This bending of light through curved space is known as *gravitational lensing*, and is depicted schematically in Figure 3. Near the Earth, space is only very weakly curved, which is why you have likely never observed the beam of a flashlight to bend. But lensing can be drastic in regions of extreme curvature, resulting in many spectacular astronomical systems, as shown in Figure 4.

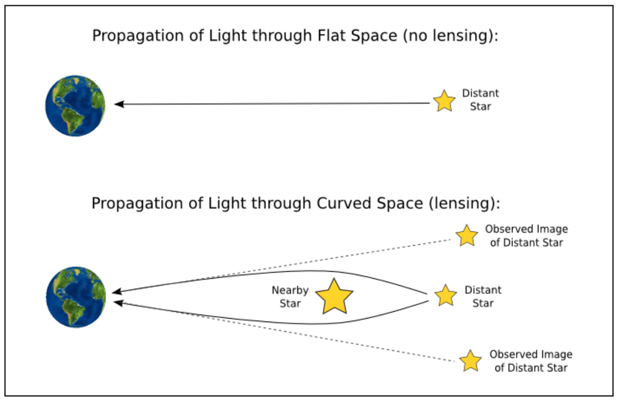

Figure 3: This figure demonstrates the observable consequences of curvature. In the upper image, light from a distant star travels directly through flat, empty space to the Earth. In the lower image, an intervening star warps space, causing the light from a distant star to follow a curved geodesic. On Earth, this light appears to originate from somewhere other than the star's true location. If the Earth, the distant star, and the intervening star are all perfectly aligned, then astronomers on Earth will observe two identical images of the distant star on either side of the intervening star. For other configurations of the two stars, the observed image will contain rings, arcs, or distorted images of the distant star. For actual images of such systems, see Figure 4. *[Courtesy of the author]*

The General Theory of Relativity

While Gauss demonstrated that the curvature of space could be measured using geometry, it was not until nearly a century later with Einstein's publication of the *General Theory of Relativity* that curved space became a major part of astronomy. Einstein suggested that the curvature of space may be governed by a law of nature which says that objects warp the space around them. He proposed an exact mathematical rule which specifies how space will curve in response to any possible arrangement of matter, roughly saying that more massive objects produce stronger curvature than lighter objects. Einstein's theory has been confirmed in numerous experiments over the last century, the first of which came just four years after its publication when the British astronomer Arthur Eddington observed that starlight bends as it passes through the curvature caused by our sun.[1]

The Shape and Contents of the Universe

Since Einstein's theory says that curvature is caused by matter, it also provides us with a clever way to determine the amount and arrangement of matter in a region of space. We measure the curvature by observing the bending of light, and then use

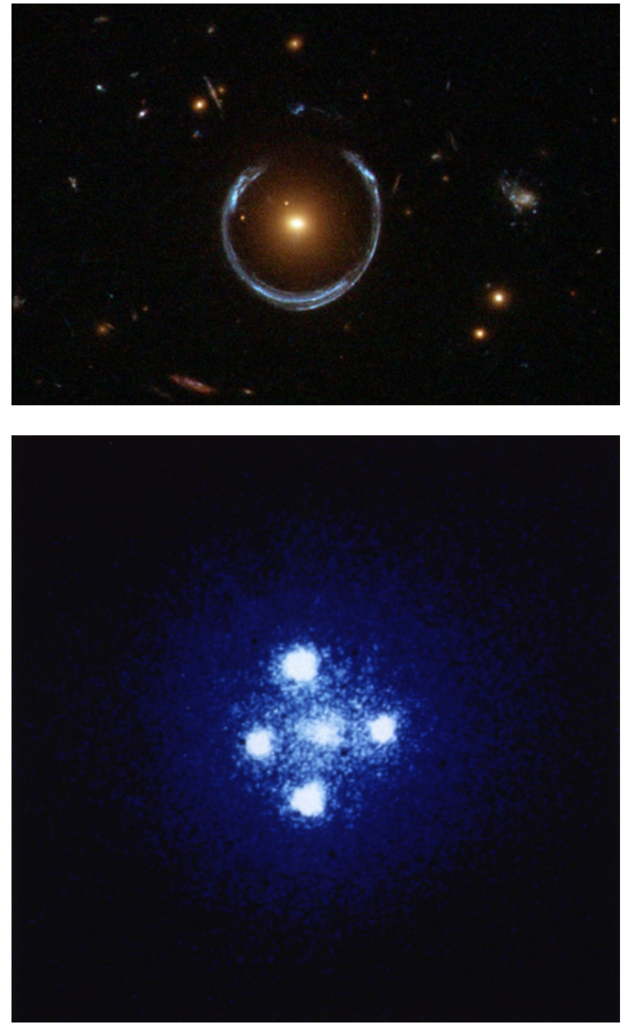

Figure 4: These images were taken by the Hubble Space Telescope, showing gravitational lensing in two different systems. In 4a, the curvature from a nearby yellowish galaxy cluster bends light from a bluish background galaxy into a large arc. And in 4b, in a system known as *Einstein's Cross*, a central galaxy has bent the light from a background source into four separate images. *[NASA, Hubblesite]*

[1] Eddington's observation is a very significant event in the history of General Relativity, though historians have disputed the legitimacy of his original measurement. To learn more about his observations, see http://news.bbc.co.uk/2/hi/science/nature/8061449.stm, and for more about his story, http://astro-geo.oxfordjournals.org/content/50/4/4.12.full

Einstein's equations to calculate which arrangement of matter would result in the measured curvature.

Galaxy clusters, as shown in Figure 5, are some of the heaviest objects in the universe and are routinely weighed by measuring the curvature they produce in space. These clusters contain up to thousands of individual galaxies, which in turn contain billions of stars each. Clusters can cause a massive warping of space, as seen by the outrageous lensing in Figure 6. This lensing is the result of light from a galaxy behind the cluster bending as it travels through the cluster on its way to Earth. When astronomers analyze the magnitude of this bending, they find something surprising: these clusters must contain much more matter than is visible in their stars and gas. This mysterious, missing matter is known as *dark matter*, and it is detected in many astronomical systems[2]. It is not yet known what exactly dark matter is composed of, and the hunt to find out is one of the major current efforts in both astronomy and physics.

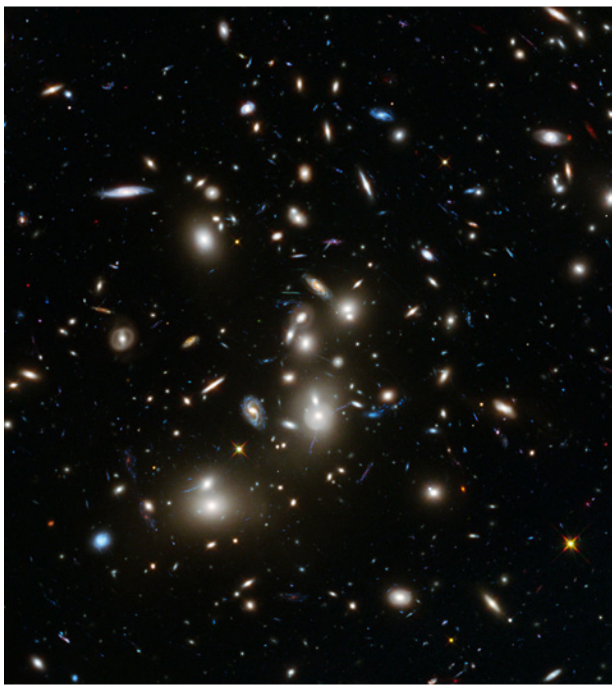

Figure 5: Abell 2744. This monstrous galaxy cluster is called *Pandora's Cluster*. Each bright shape in this image is a galaxy, containing billions of stars. *[NASA, HST]*

Figure 6: This cluster, Abell 370, exhibits dramatic lensing on the right side of the image. Numerous mild examples of lensing also appear in this image, in the form of small arcs of light around the central galaxies. To achieve this lensing, this cluster must contain vast amounts of dark matter in addition to the visible galaxies. *[NASA, ESA, HST]*

In addition to weighing individual clusters, astronomers can use curvature measurements to weigh something quite remarkable: the entire universe. As light leaves a very distant object and travels towards the Earth, it will pass through many regions of curvature during its trip across the universe and each region will cause some small bending of the light's path. The total bending that this light undergoes is a measure of the average curvature of the universe. Similar to the way in which the distortion of a background galaxy reveals the curvature of a galaxy cluster, astronomers can observe the distortion of distant *hot spots* to measure the average curvature of the universe. These hot spots are warm regions of space located at the farthest visible extent of the universe, and they are studied by observing the cosmic microwave background radiation. The radiation originating in a hot spot has a slightly different color than does radiation from elsewhere, allowing astronomers to map the size and location of these hot spots, as shown in Figure 7. The observed size of the hot spots depends on the average curvature of the universe, as depicted in Figure 8. If the universe has some average curvature, then the hot spots will either appear magnified or shrunk from their actual sizes. Since these true sizes are known from the physics of hotspots, observations such as Figure 7 can determine the magnification and hence the average curvature. And after all of this talk of curvature, astronomers found something very surprising when they observed these hot spots: they were not magni-

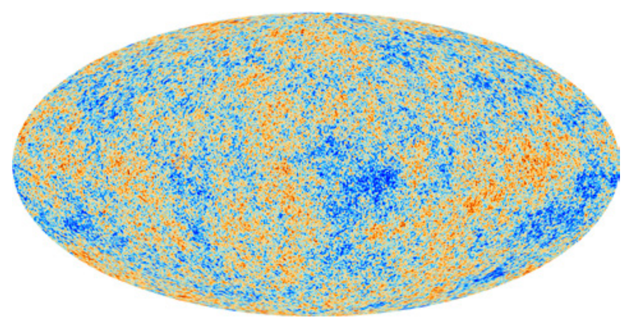

Figure 7: This is the cosmic microwave background radiation across the entire sky as revealed by the Planck satellite. The depicted color is not the direct observation, but a representation of temperature of the observed radiation. The high-temperature regions, marked in red, are the hot spots used to measure the average curvature of the universe. *[ESA, Planck]*

[2] The evidence for dark matter extends far beyond measurements of the curvature in clusters. To learn more see previous issues of *The Universe in the Classroom*, such as issue 72, Invisible Galaxies: The Story of Dark Matter, http://www.astrosociety.org/edu/publications/tnl/72/darkmatter.html

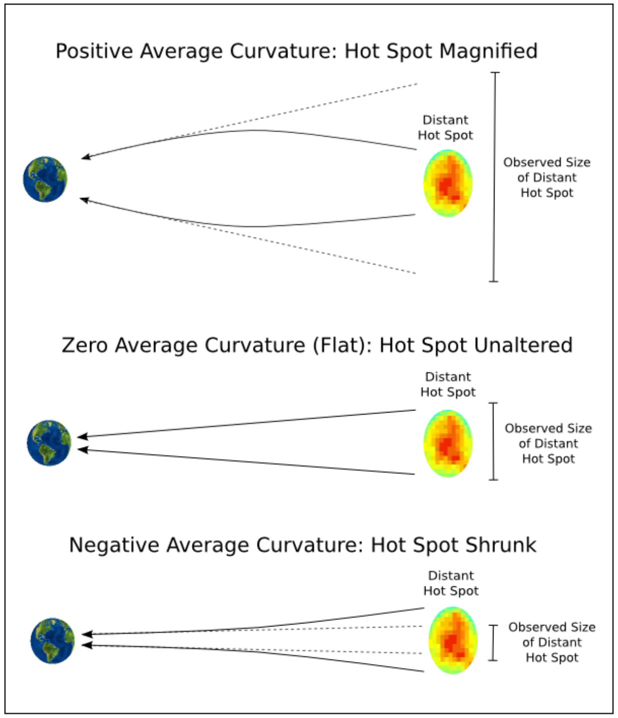

Figure 8: This is a depiction of the distortion of distant hot spots by the average curvature of the universe. The true path of light, shown in solid black, is a direct line only for a universe which is flat on average (middle diagram), while for a universe with non-zero average curvature the path of light is bent according to the sense of the curvature (top and bottom diagrams). This bending causes the hotspots to appear magnified or shrunk. Actual observations mimic the middle case of no distortion. *[Courtesy of the author]*

fied at all. Our universe is, on average, flat.

This average flatness contains yet another surprise as it allows astronomers to calculate the density of the universe's contents using Einstein's relation between curvature and matter. They find the universe contains on average 10^{-29} grams per cubic centimeter of material (this is a very small number, but remember that most of the universe is nearly empty). Yet, the total amount of matter we observe in the universe, including the dark matter discussed above, is only about 30% of this number — we are missing roughly 70% of the universe's contents. This missing material has been given the exotic name *dark energy*, and its nature is a mystery.[3] Just like dark matter, nobody knows what dark energy is, though many astronomers and physicists are currently working to find out.

Conclusion

Astronomers routinely study the curvature of space using geometric measurements. These observations reveal that our universe resembles something like an old wooden table, full of chips and scratches. Up close, you notice each little indent and defect, each bit of curvature, but if you step back and observe from a distance, those small indents are not noticeable and the table looks flat. So it is with our universe: it is mostly flat and uniform across very large distances, but if you zoom in you find it marked by small regions of strong curvature due to individual stars and galaxies. In addition to determining its shape, measurements of curvature allow astronomers to weigh the contents of the universe, revealing that familiar, visible matter accounts for only 4% of the universe's weight, as depicted in Figure 9. Understanding the mysterious other 96% motivates much current research in astronomy and physics.

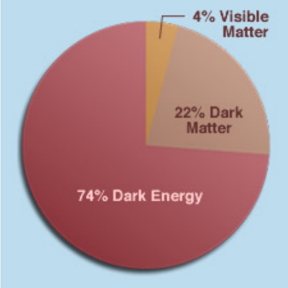

Figure 9: The contents of the universe, as revealed in part by measurements of the curvature of space. The nature of the dark 96% is still an open question. *[NASA, HST]*

Students can approach the connection between curvature and geometry through the example of two-dimensional curved surfaces, which can be explored in a hands-on fashion. They can even replicate in this way the geometric observations made by astronomers, helping them to understand these fascinating and important measurements.

Activity: Geometry of Curved Spaces

Overview

In this activity, students will explore geometry on curved surfaces and see that determining the laws of geometry is equivalent to measuring curvature. They can determine geodesics, measure the sum of the angles of triangles, and compare the circumference and diameter of circles on a curved surface to see that the *conventional* laws of geometry do not hold. This is most effective with students that have been exposed to plane geometry and will readily declare "the angles of a triangle *always* sum to 180 degrees" or "the circumference of a circle is *always* π (pi) times its diameter".

Materials

Students will need string to facilitate the measuring of distances along their surface. For the curved surface, inflatable beach balls are a good choice as they provide a large surface area and allow students to draw geometric figures directly on the balls.

- Inflatable beach balls
- Permanent Markers
- String
- Rulers
- Protractors
- Calculators

[3] Like dark matter, the evidence for dark energy also comes from multiple sources in addition to measurements of the universe's curvature. To learn more, see http://science.nasa.gov/astrophysics/focus-areas/what-is-dark-energy/ and http://hubblesite.org/hubble_discoveries/dark_energy/de-what_is_dark_energy.php

Geodesics

1. Ask students to mark two points on their beach ball at roughly the same latitude and about halfway between the equator and the poles. These points could represent two cities, one located on the West Coast of the U.S., and the other located directly to the East on the East Coast.

2. Have the students draw the path they would take to travel between their points as quickly as possible. Many students will draw a path of constant latitude.

3. Now ask them to use their string to determine the shortest path between these two points, by wrapping the string around the ball between the points as taunt as possible while still hugging the ball. This shortest path is called the *geodesic* between these points. The geodesic path from the Western point to the Eastern point is not due East, but actually arcs first NE and then SE.

4. Students can sketch their routes on a typical map projection, with lines of latitude shown as horizontal lines and lines of longitude as vertical lines (the *geographic projection*). The direct route appears as a horizontal line on this map, but the true geodesic appears as a northward arc. Have them compare their sketches to maps of airplane flight paths, as shown in Figure 10.

Triangles

1. Ask students to draw triangles on their ball by marking any three points and then connecting these points with geodesics. Some of the triangles should be large and some small.

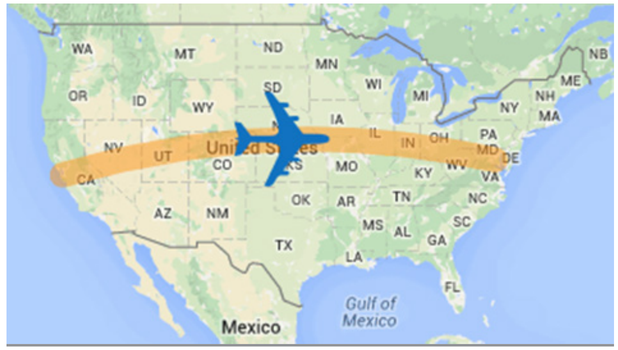

Figure 10: This is an airline flight path between San Francisco and Washington DC, plotted using a geographic projection. *[Courtesy of the author and kvikr.com.]*

2. With protractors, have the students measure the angles of their triangle and compute the sum. To measure the angles, it is helpful to press the protractor slightly into the inflatable balls.

3. Collect and display the students' sums, comparing to 180 degrees and noting any differences in the results for small and large triangles. On a *perfect* sphere, the angles of any triangle sum to greater than 180 degrees and the sum increases if the area of the triangle increases.

Circles

1. Ask students to place a dot on their ball to be the center of a circle. Ask them to determine how they would draw a circle on their ball centered on that point. A good technique is to use a piece of string as a compass: with one end held to the center, use the other end to mark the location of different points that are all the same distance from the center. After marking a number of such points, draw a smooth curve connecting them.

2. Have students draw circles, again having some large circles and some small circles.

3. Ask students how they would determine the diameter and circumference of their circles using string and a ruler. Remember that we want to consider the perspective of an ant who lives on the sphere and has no knowledge of the outside world. The appropriate diameter is the length of string along a geodesic that starts on one side of the circle, travels through the center, and ends on the opposite side of the circle.

4. Have the students measure and compute the ratio of the circumference to the diameter of their circles.

5. Gather the measurements, comparing them to π (pi) and noting the difference between large and small circles. On a *perfect* sphere, the ratio of the circumference to the diameter of any circle will be less than π (pi) and it will decrease if the area of the circle increases.

Discussion Questions

1. In what sense have you *measured* the curvature of the beach balls? How do astronomers similarly measure the curvature of the universe?

2. If light leaves a star and travels to the Earth, it usually travels a direct line towards the Earth. But, if there is another star along this line blocking its path, the light from the distant star will bend around the intervening star to reach the Earth, as in Figure 3. How is this related to the geodesics you constructed above?

3. Why do airline routes have the shape of arcs on a flat map?

4. The triangle and circle measurements become more like the familiar, flat surface results if the shapes are small. How does this compare to your experience of living on a curved Earth?

5. On a sphere such as the Earth (ignoring mountain ranges and oceans), is it possible to start walking, make exactly two turns and end up back where you started? Consider walking a triangle with one corner on the North Pole and two corners on the equator. Is this possible on a flat surface, such as a tabletop?

Ideas for Older Students

Secretly Flat Surfaces

Some surfaces look curved but are actually geometrically flat, i.e. their laws of geometry are identical to those for a flat surface. The outside of a cone or the outer surface of a cylinder are examples. Students can see this by using a traffic cone or food can for the above measurements. If an ant trapped on one of these surfaces were to use geometric measurements to measure the curvature, what would they find?

You can also see that these surfaces have flat geometry by the fact that they can be created by rolling a flat piece of paper without tearing it — any curves and angles drawn while the paper is unrolled will maintain their size and shape after it is rolled. This cannot be done to make a truly curved surface, like a sphere. Obviously, cylinders and cones are different from a flat plane in some way, because an ant walking on them can walk in a straight line without turning and eventually return to its starting position. But this is due to a difference in something other than curvature: mathematicians say that these surfaces differ *topologically*, but are identical *geometrically*.

Non-uniform Curvature

A sphere is a special surface since it has *uniform* curvature, meaning the laws of geometry do not depend on where you are on the surface. This is in contrast to something like a football, where the geometry near the tip will be different than in the flatter, middle portion. Students can explore this by using non-uniformly curvature objects for the above measurements, such as footballs, flower vases, or saddles.

Since the galaxies in our universe are distributed roughly uniformly in space, Einstein's equations predict that they produce an average curvature which is uniform across the universe.

Activity: Measuring the Radius of the Earth with Eratosthenes

Eratosthenes's measurement of the radius of the Earth is an incredible example of the power of careful observation. His method involves measuring the position of the sun at the same time from two different locations on the Earth. The measurement itself is straightforward, though it will require collaboration with another group of students located several hundred miles North or South of you. The coordination is not too involved, as the two groups simply need to make the same measurement at the same time and then share results. This is a great opportunity to demonstrate and practice collaboration, which is an essential part of all contemporary science.

The Eratosthenes Project provides resources to help classrooms around the globe perform Eratosthenes's measurement. For more information, activity guides, and collaboration resources see their website at http://www.eaae-astronomy.org/eratosthenes/